

Solution Sheet 3

Exercise 1. Let G be a Hausdorff topological group acting continuously on a Hausdorff space X . Let

$$\rho : G \times X \rightarrow X \times X, \quad \rho(g, x) = (gx, x).$$

For a compact $K \subset X$, put

$$G_K = \{g \in G \mid gK \cap K \neq \emptyset\}.$$

- (1) Show that $\rho(G \times X)$ is closed if and only if X/G is Hausdorff.
- (2) Show that ρ is proper if and only if G_K is compact for every compact $K \subset X$.
Hint: Take $L \subset X \times X$ compact and set $K_1 = \pi_1(L)$, $K_2 = \pi_2(L)$. Show that $\rho^{-1}(L) \subset G_{K_1 \cup K_2} \times K_2$.
- (3) Suppose G is discrete. Show that ρ is proper if and only if G_K is finite for every compact $K \subset X$.

Solution 1. (1) We use the fact that a topological space X is Hausdorff if and only if its diagonal $\Delta_X = \{(x, x)\} \subset X \times X$ is closed. Let $q : X \rightarrow X/G$ be the quotient map. It is an open map, hence $q \times q$ is also open, and thus a quotient map. Therefore a subset $A \in X/G \times X/G$ is closed if and only if $(q \times q)^{-1}(A)$ is closed. Consequently,

$$\begin{aligned} X/G \text{ is Hausdorff} &\iff \Delta_{X/G} \text{ is closed} \\ &\iff (q \times q)^{-1}(\Delta_{X/G}) \text{ is closed.} \end{aligned}$$

We conclude by noting that

$$\begin{aligned} (q \times q)^{-1}(\Delta_{X/G}) &= \{(x, y) : \exists g \in G \text{ s.t. } x = gy\} \\ &= \{(gx, x) : g \in G, x \in X\} = \rho(G \times X). \end{aligned}$$

- (2) Suppose that ρ is proper. Let $K \subset X$ be a compact subset. Then

$$\rho^{-1}(K \times K) = \{(g, x) \in G \times X : x \in K, gx \in K\}$$

is compact. Its projection to G is exactly $\{g \in G : \exists x \in K \text{ with } gx \in K\} = G_K$, which is therefore also compact.

Conversely, suppose that G_K is compact for all compact subsets $K \subset X$. Let $L \subset X \times X$ be a compact subset and let $K_1 = \pi_1(L)$ and $K_2 = \pi_2(L)$ (here π_1 and π_2 are the coordinate projections). Both K_1 and K_2 are compact. If $(g, x) \in \rho^{-1}(L)$ we have $(gx, x) \in L$, hence $gx \in K_1$ and $x \in K_2$. Hence $gx \in gK_2 \cap K_1$, which means that

$$g \in G_{K_1, K_2} := \{g \in G : gK_2 \cap K_1 \neq \emptyset\} \subset G_{K_1 \cup K_2}.$$

Therefore it follows that

$$\rho^{-1}(L) \subset G_{K_1 \cup K_2} \times K_2.$$

By hypothesis, the product $G_{K_1 \cup K_2} \times K_2$ is compact. The compact set L is closed (X is Hausdorff). Therefore $\rho^{-1}(L)$ is closed in $G \times X$, and hence closed in the subspace $G_{K_1 \cup K_2} \times K_2$. A closed subspace in a compact space is compact, hence $\rho^{-1}(L)$ is compact.

- (3) This follows from the previous result, since compact subsets of a discrete topological space are finite.

Exercise 2. Give an example of a non-proper action of \mathbb{Z} on the complex plane by biholomorphisms.

Solution 2. We define a holomorphic action of \mathbb{Z} on the complex plane by $n \cdot z = 2^n z$ for all $n \in \mathbb{Z}$ and $z \in \mathbb{C}$. We note that 0 is fixed by all iterates. Let $K = \overline{\mathbb{D}}$ be the compact unit disk. Then $n \cdot K \cap K \neq \emptyset$ for all $n \in \mathbb{Z}$. Hence $G_K = \mathbb{Z}$, which is not finite. From the criterion in exercise 1, we deduce that the action is not proper.

Exercise 3. Let $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$ and $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$ act on \mathbb{H} by

$$z \mapsto \gamma z = \frac{az + b}{cz + d} \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let

$$\mathcal{F} = \left\{ z \in \mathbb{H} : -\frac{1}{2} \leq \Re z \leq \frac{1}{2}, |z| \geq 1 \right\}.$$

- (1) Show that \mathbb{H} is preserved by the action of $\mathrm{PSL}_2(\mathbb{Z})$.
- (2) If $z_1, z_2 \in \overline{\mathcal{F}}$ are distinct and $z_2 = \gamma z_1$ for some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, prove that either $\Re z_1 = \pm \frac{1}{2}$ and $z_2 = z_1 \pm 1$, or $|z_1| = 1$ and $z_2 = -1/z_1$.
- (3) Deduce that \mathcal{F} contains exactly one representative of each $\mathrm{PSL}_2(\mathbb{Z})$ -orbit (up to boundary points). Such a \mathcal{F} is called a fundamental domain.
- (4) Show that the action of $\mathrm{PSL}_2(\mathbb{Z})$ on \mathbb{H} is proper.

Solution 3. (1) For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z})$ we have

$$\Im \left(\frac{az + b}{cz + d} \right) = \frac{ad - bc}{|cz + d|^2} \Im(z) = \frac{1}{|cz + d|^2} \Im(z),$$

which shows that if $\Im(z) > 0$, then also $\Im(\gamma z) > 0$. Thus \mathbb{H} is preserved by $\mathrm{PSL}_2(\mathbb{Z})$ (in fact by $\mathrm{PSL}_2(\mathbb{R})$).

- (2) The key observation is that for all $z \in \overline{\mathcal{F}}$ and any coprime integers (c, d) we have $|cz + d| \geq 1$, with equality only if either $c = 0$ and $|\Re(z)| = \frac{1}{2}$, or $|z| = 1$. We prove this observation later (TODO). Let $z_1 \in \overline{\mathcal{F}}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z})$ with $z_2 = \gamma z_1 \in \overline{\mathcal{F}}$. Then

$$\Im(z_2) = \Im(\gamma z_1) = \frac{\Im(z_1)}{|cz_1 + d|} \leq \Im(z_1),$$

and similarly $\Im(z_1) \leq \Im(z_2)$. We deduce that $|cz_1 + d| = 1$. By the observation, this forces either

- (a) $c = 0$ and $|\Re(z_1)| = \frac{1}{2}$. Then $ad = 1$, hence $a = d = \pm 1$. So in $\mathrm{PSL}_2(\mathbb{Z})$ we may take $\gamma = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ and then $z_2 = z_1 + b$. Since $z_1, z_2 \in \overline{\mathcal{F}}$ are distinct, we must have $b = \pm 1$. Thus $z_2 = z_1 \pm 1$.
- (b) $|z_1| = 1$. Then $|c| = 1$ and $d = 0$. The determinant condition gives $b = -c$. Therefore in $\mathrm{PSL}_2(\mathbb{Z})$ we can take $\gamma = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}$ and so $z_2 = a - 1/z_1$. Using $1/z_1 = \overline{z_1}$ gives $\Re(z_2) = a - \Re(z_1)$. Thus

$$a - \Re(z_1) \in [-1/2, 1/2] \quad \text{with } \Re(z_1) \in [-1/2, 1/2].$$

This forces $a = 0$ (except at the two corner points already handled). Thus on the circular boundary we get the pairing $z_2 = -1/z_1$.

- (3) Fix $z \in \mathbb{H}$. Choose a primitive integer pair (c, d) that is minimal among $\{|cz + d| : (c, d) = 1\}$. Then choose $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z})$ and let $w = \gamma z$. Translating horizontally preserves the denominator and its minimality, thus we may assume

that $|\Re(w)| \leq \frac{1}{2}$. Since (a, b) is also primitive, minimality of $|cz + d|$ implies that $|az + d| \geq |cz + d|$. Hence

$$|w| = \frac{|az + b|}{|cz + d|} \geq 1.$$

Therefore $w \in \mathcal{F}$. With part (2), this means that \mathcal{F} meets each $\mathrm{PSL}_2(\mathbb{Z})$ -orbit in exactly one point up to boundary pairings; hence $\overline{\mathcal{F}}$ is the closure of a fundamental domain.

(4) We use the criterion from exercise 1. Let K be a compact subset of \mathbb{H} . Put

$$m = \min_{z \in K} \Im z > 0, \quad M = \max_{z \in K} \Im z, \quad R = \max_{z \in K} |z|.$$

If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z})$ satisfies $\gamma K \cap K \neq \emptyset$, pick $z_0 \in K$ with $w_0 = \gamma z_0 \in K$.

Using $\Im(\gamma z) = \Im z / |cz + d|^2$ we get

$$|cz_0 + d|^2 = \frac{\Im z_0}{\Im w_0} \leq \frac{M}{m}.$$

Hence

$$|c| \leq \frac{|cz_0 + d|}{\Im z_0} \leq \frac{\sqrt{M/m}}{m},$$

$$|d| \leq |cz_0 + d| + |c||z_0| \leq \sqrt{M/m} + \frac{\sqrt{M/m}}{m} R,$$

so c, d range over finite sets. Apply the same reasoning to $\gamma^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ at w_0 :

$$|-cw_0 + a|^2 = \frac{\Im w_0}{\Im z_0} \leq \frac{M}{m} \Rightarrow |a| \leq \sqrt{M/m} + \frac{\sqrt{M/m}}{m} R,$$

so a ranges also over a finite set. Finally, if $c \neq 0$ then $b = (ad - 1)/c$ is finite. If $c = 0$ then $a = d = \pm 1$ and $w_0 = z_0 + \frac{b}{a} \in K$ forces $|b| = |\Re(w_0) - \Re(z_0)|$, which is bounded on K . Thus only finitely many γ satisfy $\gamma K \cap K \neq \emptyset$, i.e. the action is proper.

Exercise 4. (for credit, due on 5 October) Let $\omega, \omega' \in \mathbb{H}$ and $\Lambda = \mathbb{Z} + \omega\mathbb{Z}$, $\Lambda' = \mathbb{Z} + \omega'\mathbb{Z}$. Let $\pi : \mathbb{C} \rightarrow \mathbb{C}/\Lambda$ and $\pi' : \mathbb{C} \rightarrow \mathbb{C}/\Lambda'$ be the quotient maps.

(0) (0.5 points) Let $\mu(\Lambda) = \inf\{|\lambda| : \lambda \in \Lambda \setminus \{0\}\}$. Note that $\mu(\Lambda) > 0$. For any $r < \mu(\Lambda)/2$ and any $z_0 \in \mathbb{C}$, show that the restriction

$$\pi|_{B_r(z_0)} : B_r(z_0) \rightarrow \mathbb{C}/\Lambda$$

is injective.

- (1) (0.5 points) Show that \mathbb{C}/Λ is homeomorphic to the torus $S^1 \times S^1$.
- (2) (0.5 points) Endow \mathbb{C}/Λ with a structure of a Riemann surface.
- (3) (2 points) Show that any holomorphic map $f : \mathbb{C}/\Lambda \rightarrow \mathbb{C}/\Lambda'$ lifts to an affine map $F(z) = az + b$ with $a\Lambda \subset \Lambda'$. **Hint:** Lift f to a holomorphic map F , whose derivative is Λ -periodic, and apply Liouville's theorem.
- (4) (1.5 points) Show that \mathbb{C}/Λ is biholomorphic to \mathbb{C}/Λ' if and only if there exist $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$ such that $\omega' = \frac{a\omega + b}{c\omega + d}$.

Solution 4. (0) Suppose $\pi(z_1) = \pi(z_2)$, that is, $z_1 - z_2 \in \Lambda$. Assume that $z_1, z_2 \in B_r(z_0)$. Then by the triangle inequality we get

$$|z_1 - z_2| \leq |z_1 - z_0| + |z_2 - z_0| < 2r < \mu(\Lambda).$$

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- But if $z_1 - z_2 \in \Lambda \setminus \{0\}$, then $|z_1 - z_2| \geq \mu(\Lambda)$, a contradiction. Therefore $z_1 = z_2$.
- (1) The quotient \mathbb{C}/Λ identifies points in \mathbb{C} that differ by an element of Λ . Topologically, this is the quotient of \mathbb{R}^2 by the subgroup generated by the two independent vectors corresponding to 1 and w . This subgroup is isomorphic to \mathbb{Z}^2 , and the quotient $\mathbb{R}^2/\mathbb{Z}^2$ is homeomorphic to the torus $S^1 \times S^1$.
 - (2) Part (0) guarantees that for each point in \mathbb{C}/Λ , there exists a neighborhood U such that $\pi^{-1}(U)$ contains a disk $D \subset \mathbb{C}$ on which π is injective. We use $(\pi|_D)^{-1} : U \rightarrow \mathbb{C}$ as a coordinate chart. The transition maps between coordinate charts are translations of the form $z \mapsto z + \lambda$ with $\lambda \in \Lambda$. Optional: the action of Λ by translations is free and proper, so π is a holomorphic covering.
 - (3) Let $f : \mathbb{C}/\Lambda \rightarrow \mathbb{C}/\Lambda'$ be holomorphic. We can lift f to a holomorphic map $F : \mathbb{C} \rightarrow \mathbb{C}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{F} & \mathbb{C} \\ \pi \downarrow & & \downarrow \pi' \\ \mathbb{C}/\Lambda & \xrightarrow{f} & \mathbb{C}/\Lambda' \end{array}$$

The existence of this lift follows from covering space theory: Set $g := f \circ \pi : \mathbb{C} \rightarrow \mathbb{C}/\Lambda'$. Since \mathbb{C} is simply connected and π' a covering map, the lifting theorem for covering spaces implies the following: for any $z_0 \in \mathbb{C}$ and any $w_0 \in \mathbb{C}$ with $\pi'(w_0) = g(z_0)$, there exists a unique continuous lift $F : \mathbb{C} \rightarrow \mathbb{C}$ such that $\pi' \circ F = g$ and $F(z_0) = w_0$. To see that F is also holomorphic, fix $z \in \mathbb{C}$. Because π and π' are local biholomorphisms, there exist small disks $U \ni z$ and $V \ni F(z)$ such that $\pi|_U : U \rightarrow \pi(U)$ and $\pi'|_V : V \rightarrow \pi'(V)$ are biholomorphisms. On U , the restriction of F

$$F|_U = (\pi'|_V)^{-1} \circ f \circ (\pi|_U)$$

is a composition of holomorphic maps, hence holomorphic on U . Because z was arbitrary, F is holomorphic on \mathbb{C} . From the commutative diagram

$$\pi'(F(z)) = f(\pi(z))$$

we see that $F(z + \lambda)$ and $F(z)$ map to the same point in \mathbb{C}/Λ' , i.e.

$$F(z + \lambda) - F(z) \in \Lambda' \quad \text{for all } \lambda \in \Lambda.$$

Since this difference is a holomorphic function taking values in the discrete set Λ' , it must be constant. Differentiating both sides with respect to z implies that $F'(z + \lambda) - F'(z) = 0$, i.e. $F'(z)$ is Λ -periodic. Since F' is a holomorphic and Λ -periodic function, it is bounded on a fundamental parallelogram for Λ (a continuous function on a compact set is bounded), and hence bounded on all of \mathbb{C} . Consequently, by Liouville's theorem, it is constant. Therefore there exists $a \in \mathbb{C}$ such that $F'(z) = a$, implying $F(z) = az + b$ for some $b \in \mathbb{C}$. This then gives

$$F(z + \lambda) - F(z) = a\lambda \in \Lambda' \quad \text{for all } \lambda \in \Lambda.$$

We conclude that $a\Lambda \subset \Lambda'$.

- (4) Let $g : \mathbb{C}/\Lambda \rightarrow \mathbb{C}/\Lambda'$ be a biholomorphism. It lifts to an affine transformation $F(z) = az + \beta$ for some $\alpha \in \mathbb{C} \setminus \{0\}$. Moreover, its inverse is also an affine transformation: $F^{-1}(z) = \frac{1}{\alpha}z + \gamma$. Thus $\alpha\Lambda \subseteq \Lambda'$ and $\frac{1}{\alpha}\Lambda' \subseteq \Lambda$, which yields $\alpha\Lambda = \Lambda'$. This means that $(1, \omega')$ and $(\alpha, \alpha\omega)$ are two \mathbb{Z} -bases of Λ' with the same orientation. Therefore, there exists a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ such that

$$\begin{pmatrix} \omega' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha\omega \\ \alpha \end{pmatrix}.$$

From this we deduce that $\omega' = \frac{a\omega+b}{c\omega+d}$. Conversely, suppose that $\omega' = \frac{a\omega+b}{c\omega+d}$ holds. We set

$$\alpha = \frac{1}{c\omega+d} \in \mathbb{C}^\times;$$

we have $\alpha(1, w) = (\alpha, \alpha w)$. Then we get

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha\omega \\ \alpha \end{pmatrix} = \begin{pmatrix} \alpha(a\omega+b) \\ \alpha(c\omega+d) \end{pmatrix} = \begin{pmatrix} \omega' \\ 1 \end{pmatrix}.$$

Therefore the vector $(\alpha, \alpha w)$ is mapped by an invertible integer matrix to the basis $(1, w')$ of $\Lambda' = \langle 1, w' \rangle$. This means that $(\alpha, \alpha w)$ is another basis of the \mathbb{Z} -module Λ' . Hence $\Lambda' = \alpha\Lambda$. The map $z \mapsto \alpha z$ induces a biholomorphism $\mathbb{C}/\Lambda \cong \mathbb{C}/\Lambda'$.

Exercise 5. (*)

For $\Im w > 0$ we set $\Lambda = \mathbb{Z} + w\mathbb{Z}$ and $E_w = \mathbb{C}/\Lambda$.

- (1) Show that if there exists $a \in \mathbb{C} \setminus \mathbb{R}$ with $a\Lambda \subset \Lambda$, then w is imaginary quadratic (algebraic of degree 2 over \mathbb{Q}).
- (2) Conversely, if w is imaginary quadratic, construct a non-real a with $a\Lambda \subset \Lambda$.
- (3) Define $\text{End}(E_w) = \{a \in \mathbb{C} : a\Lambda \subset \Lambda\}$. Deduce that either $\text{End}(E_w) = \mathbb{Z}$ or

$$\text{End}(E_w) = \{n + ma : n, m \in \mathbb{Z}\}$$

for some non-real a as above.

Solution 5. (1) The condition $a\Lambda \subset \Lambda$ means that the images of the generators 1 and w land back in Λ , so

- there exists unique $n, m \in \mathbb{Z}$ with $a = n + mw$,
- there exists unique $p, q \in \mathbb{Z}$ with $aw = p + qw$.

Consequently w satisfies the quadratic equation

$$(n + mw)w = p + qw \Rightarrow nw^2 + (m - q)w - p = 0.$$

Because the coefficients of this quadratic equation are real, the other root is \bar{w} . The discriminant of the quadratic equation is then

$$\Delta = n^2(w - \bar{w})^2 = -4n^2(\Im w)^2 < 0,$$

which is not a square in \mathbb{Q} . Hence the polynomial is irreducible over \mathbb{Q} and w has degree 2. **Remark:** If $a \in \mathbb{R}$ and $a = n + mw$, then $m = 0$ (since otherwise $w \in \mathbb{R}$) and hence $a = n \in \mathbb{Z}$. Therefore real endomorphisms are just integer multiplications on \mathbb{C}/Λ ; a non-real a gives new endomorphisms.

- (2) ...
- (3) ...